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ARTIGO ORIGINAL

ANÁLISE DAS FUNÇÕES DE ENERGIA E ONDA E DAS PROPRIEDADES TERMODINÂMICAS DA EQUAÇÃO DE SCHRODINGER 6-DIMENSIONAL SOB DUPLO OSCILADOR EM FORMA DE ANEL (DRSO) E POTENCIAIS MANNING-ROSEN USANDO MÉTODO SUSY QM

ANALYSIS OF ENERGY AND WAVE FUNCTIONS AND THE THERMODYNAMICS PROPERTIES OF THE 6-DIMENSIONAL SCHRODINGER EQUATION UNDER DOUBLE RING-SHAPE OSCILLATOR (DRSO) AND MANNING-ROSEN POTENTIALS USING SUSY QM METHOD

ANALISIS ENERGY DAN FUNGSI GELOMBANG DAN SIFAT TERMODINAMIS DARI PERSAMAAN SCHRODINGER 6-DIMENSI DIBAWAH PENGARUH POTENSIAL DOUBLE RING-SHAPE OSCILLATOR (DRSO) DAN POTENSIAL MANNING-ROSEN MENGGUNAKAN METODE SUSY QM.

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Received 17 July 2020; received in revised form 18 August 2020; accepted 25 October 2020

RESUMO

As soluções exatas das equações de Schrodinger (SE) no sistema de coordenadas D-dimensional têm atraído a atenção de muitos pesquisadores teóricos nos ramos da física quântica e química quântica. Os autovalores de energia e a função de onda são as soluções da equação de Schrodinger que implicitamente representam o comportamento de um sistema mecânico quântico. O estudo teve como objetivo obter os autovalores, funções de onda e propriedades termodinâmicas da equação de Schrodinger 6-Dimensional sob duplo oscilador em forma de anel (DRSO) e potencial de Manning-Rosen. O método de separação variável foi aplicado para reduzir a única equação de Schrodinger 6-Dimensional dependente do potencial radial e angular não central em cinco equações de Schrodinger unidimensionais: uma equação de Schrodinger radial e cinco equações de Schrodinger angulares. Cada uma dessas equações de Schrodinger unidimensionais foi resolvida usando o método SUSY QM para obter um autovalor e uma função de onda da parte radial, cinco autovalores e cinco funções de onda angular da parte angular. Algumas propriedades termodinâmicas, tais como energia vibracional média *U*, calor específico vibracional *C*, energia livre vibracional *F* e entropia vibracional *S*, foram obtidas por meio das equações de energia radial. Os resultados mostraram que, exceto o n_{l1}, todo incremento do número quântico angular diminui os valores de energia. Incrementos de todos os parâmetros potenciais aumentam os valores de energia. O incremento do número quântico angular e do parâmetro de potenciais aumenta a amplitude e desloca as funções de onda para a esquerda. Entretanto, o incremento de n_{l1}, *α, σ* e *ρ* diminui a amplitude e muda as funções de onda para a direita. Além disso, a energia média vibracional *U* e a energia livre *F* aumentaram com o aumento do valor dos parâmetros dos potenciais, onde o parâmetro ω tem o efeito dominante do que os outros parâmetros. O calor vibracional específico *C* e a entropia *S* são afetados apenas pelo parâmetro *ω*, onde *C* e *S* diminuem com o aumento de *ω*.

Palavras-chave: *Sistema D-dimensional, potencial não central, potencial de forma de anel, supersimetria, mecânica quântica, quantidades termodinâmicas.*

ABSTRACT

The exact solutions of the Schrodinger equations (SE) in the D-dimensional coordinate system have attracted the attention of many theoretical researchers in branches of quantum physics and quantum chemistry. The energy eigenvalues and the wave function are the solutions of the Schrodinger equation that implicitly represents the behavior of a quantum mechanical system. This study aimed to obtain the eigenvalues, wave functions, and thermodynamic properties of the 6-Dimensional Schrodinger equation under Double Ring-Shaped Oscillator (DRSO) and Manning-Rosen potential. The variable separation method was applied to reduce the one

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DOI: 10.52571/PTQ.v17.n36.2020.580_Periodico36_pgs_565_583.pdf

6-Dimensional Schrodinger equation depending on radial and angular non-central potential into five one-dimensional Schrodinger equations: one radial and five angular Schrodinger equations. Each of these one-dimensional Schrodinger equations was solved using the SUSY QM method to obtain one eigenvalue and one wave function of the radial part, five eigenvalues, and five angular wave functions angular part. Some thermodynamic properties such, the vibrational mean energy U, vibrational specific heat C, vibrational free energy F, and vibrational entropy S, were obtained using the radial energy equations. The results showed that except the n_{l1}, all increment of angular quantum number decreases the energy values. Increments of all potential parameter increase the energy values. Increment of angular quantum number and potentials parameter increases the amplitude and shifts the wave functions to the left. However, the increment of n_{l1}, α, σ, and ρ decrease the amplitude and shift wavefunctions to the right. Moreover, the vibrational mean energy U and free energy F increased as the increasing value of potentials parameters, where the ω parameter has the dominant effect than the other parameters. The vibrational specific heat C and entropy S affected only by the ω parameter, where C and S decreased as the increase of ω.

Keywords: *D-dimensional system*, *non-central potential*, *ring shape potential*, *supersymmetry quantum mechanics*, *thermodynamics quantities*.

ABSTRAK

Solusi eksak dari persamaan Schrodinger (SE) dalam sistem koordinat D-dimensi telah menarik banyak perhatian peneliti-peneliti fisika teori dalam cabang fisika dan kimia kuantum. Nilai eigen energi dan fungsi gelombang merupakan solusi dari persamaan Schrodinger, yang secara implisit merepresentasikan perilaku suatu sistem kuantum. Penelitian ini bertujuan untuk mendapatkan nilai eigen, fungsi gelombang, dan sifat termodinamika dari persamaan Schrodinger 6 Dimensi dengan potensial *Double Ring-Shaped Oscillator* (DRSO) dan Manning-Rosen. Metode pemisahan variabel diterapkan untuk mereduksi satu persamaan Schrodinger 6-Dimensi yang bergantung pada potensial non-sentral fungsi radial dan sudut menjadi lima persamaan Schrodinger satu dimensi: satu persamaan Schrodinger bagian radial, dan lima persamaan Schrodinger bagian sudut. Masing-masing persamaan Schrodinger satu dimensi ini diselesaikan menggunakan metode SUSY QM untuk mendapatkan: satu nilai eigen dan satu fungsi gelombang dari bagian radial, lima nilai eigen dan lima fungsi gelombang sudut dari bagian sudut. Beberapa sifat termodinamika seperti vibrasi energi rata-rata U, vibrasi panas spesifik C, vibrasi energi bebas F, dan vibrasi entropi S diperoleh dengan menggunakan persamaan energi radial. Hasil penelitian menunjukkan bahwa, selain n_{l1}, semua kenaikan bilangan kuantum sudut menurunkan nilai energi. Kenaikan semua parameter potensial meningkatkan nilai energi. Penambahan bilangan kuantum sudut dan parameter potensial meningkatkan amplitudo dan menggeser fungsi gelombang ke kiri. Namun, kenaikan n_{l1}, α, σ, dan ρ menurunkan amplitudo dan menggeser fungsi gelombang ke kanan. Selain itu, nilai vibrasi mean energi U dan energi bebas F semakin meningkat seiring dengan meningkatnya nilai parameter potensial, dimana parameter ω memiliki pengaruh yang dominan dibandingkan dengan parameter lainnya. vibrasi panas jenis C dan entropi S hanya dipengaruhi oleh parameter ω, di mana C dan S menurun seiring bertambahnya ω.

Kata Kunci: *Sistem D-Dimensi*, *potensial non sentral*, *potensial cincin ganda*, *mekanika kuantum super simetri*, *kuantitas termodinamika*.

1. INTRODUCTION:

The solution of the Schrodinger equation (SE), Klein Gordon (KGE) and Dirac equations (DE) have been attracted the attention of many researchers in theoretical physics area (Durmus and Yasuk, 2007; Karayer, 2019; Yahya and Oyewumi, 2016) because these equations contain all information on the quantum mechanics system. The SE is used to analyze the quantities of nonrelativistic systems, while the KGE and DE are used in relativistic systems to analyze the spin-0 and spin-$\frac{1}{2}$ particles, respectively (Arda and Sever, 2009; Candemir, 2016; Zarrinkamar *et al*., 2010). The solution of SE, KGE, and DE with some potentials have a great rule in atomic and particles physics, plasma and solid-state (Ebomwonyi, *et al*., 2017; Zhang, 2000), scattering cross-section, tunneling, and decay rate (Antia et al., 2017). However, to get analytic solutions from these equations requires special methods and approaches. Various methods are still being developed in the interest of looking for nonrelativistic wave solutions. The Nikiforov-Uvarov (NU) method has been used to solve the Dirac Equation, which is influenced by hyperbolic potential (Karayer, 2019), Klein Gordon Equation (KGE) with multi-parameter q-deformed Wood Saxon potential (Lütfüoğlu, *et al*., 2018). Asymptotic Iteration Method (AIM) has been used to solve several problems. For instance, this method has been used to investigate Dirac Equation with Morse potential with tensor interaction (Alsadi, 2015), and Klein Gordon Equation with exponential scalar and vector

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potential (Ikhdair and Falaye, 2013), relativistic and nonrelativistic wave equation under Poschl-Teller potential and its thermodynamic properties (Taşkın, *et al*., 2008). Romanovski polynomial method has been used to investigate a quantum mechanical system of Dirac with the effect of Scarf plus new tensor coupling potential (Suparmi, *et al*., 2014). The quantization rule approach has been used to solve the Schrodinger Equations problem with hyperbolic plus second Poschl-Teller potential (Dong and Gonzalez-Cisneros, 2008). Supersymmetry Quantum Mechanics (SUSY QM) method to analyze the Schrodinger equation with heavy-quarkonia potential (Abu-Shady and Ikot, 2019) and Dirac equation under hyperbolic and Coulomb potential (Hassanabadi, *et al*., 2011).

$$V(r,\theta)=\frac{1}{2}\mu\omega^2r^2+\frac{\hbar^2}{2\mu}\left(\frac{\alpha}{r^2\sin^2\theta}+\frac{\sigma}{r^2\cos^2\theta}\right)$$
(Eq. 1)

The Double Ring-Shape Oscillator (DRSO) potential (1) is a type of ring-shaped potential with its mathematical interests and its physics and chemistry quantum applications. The ring-shaped type potential is one of non-central potential with a highly symmetrical system because of its invariance under reflection and axial symmetry (Carpio-Bernido and Chrisopher, 1989). This ring-shape type potential is generally a combination of Coulomb, oscillator, or Hartman potential which involves $\left(\frac{1}{r^2\sin^2\theta}\right)$ on a ring-shaped type of single-ring-shaped and $\left(\frac{1}{\sin^2\theta}+\frac{1}{\cos^2\theta}\right)$ in the double ring-shaped type (You, *et al*, 2018). This potential type in recent times has been an interesting topic of researchers in physics and quantum chemistry because of its importance when applied to describe the structure of benzene molecules in quantum chemistry and deformed nuclei in nuclear physics. Ring-shaped harmonic oscillators are also used to study the spin symmetry of an antinucleon embedded in nucleus and the linear and nonlinear optical effects of moving electrons in the non-central field (Chang-Yuan, *et al*, 2013; Fa-Lin, Lu and Chang-Yuan, 2010; Hassanabadi, *et al*, 2014; Ikot, A. N., *et al*, 2016; Sun, *et al*, 2014; Yasuk and Durmus, 2007; You *et al.*, 2018).

Improvement to solve the Schrodinger equation in a higher dimensional system still necessary to carry out because it is important under the field of quantum physics (André, *et al.*, 2019; Falaye and Oyewumi, 2011; Onate, *et al.*, 2018; Wang, *et al.*, 2002). One of the motivations that attracted the attention of scientists to continually identify D-dimensional systems is the pretentious unified theory 20th century between the Relativity and Quantum theory. Another reason is to obtain clarity from one of the products from string and supergravity theory, the Klauza-Klein theory if its additional dimensions are the spatial dimension (Dong, 2011).

Therefore, to obtain more advancement information on the D-dimensional quantum mechanical system, it should not be restricted to four or five-dimensional spaces. The investigation needs to be made for higher dimensional spaces and various potential systems to get more general solutions. So, this study tried to analyze the 6-dimensional quantum mechanical system, both for radial and each of its angular parts.

$$V(\theta)=\frac{\hbar^2}{2\mu}\frac{\upsilon(\upsilon+1)}{\sin\theta}-\frac{\hbar^2}{2\mu}\frac{2\rho}{\upsilon}\cot\theta$$
(Eq. 2)

DRSO (1) plus Manning Rosen potential (2) could be extended into a 6-Dimensional separable non-central potential written as follows

$$V(r,\theta_1,\theta_2,\theta_3\theta_4,\theta_5)=V(r)+\frac{1}{r^2}\left\{\frac{V_1(\theta_1)}{\sin^2\theta_2\sin^2\theta_3\sin^2\theta_4\sin^2\theta_5}+\frac{V_2(\theta_2)}{\sin^2\theta_3\sin^2\theta_4\sin^2\theta_5}+\frac{V_3(\theta_3)}{\sin^2\theta_4\sin^2\theta_5}+\frac{V_4(\theta_4)}{\sin^2\theta_5}+V_5(\theta_5)\right\}$$
(Eq. 3)

with

$$V(r)=\frac{1}{2}\mu\omega^2r^2$$
(Eq. 4)

$$V_1(\theta_1)=\frac{\hbar^2}{2\mu}\left(\frac{\alpha}{\sin^2\theta_1}+\frac{\sigma}{\cos^2\theta_1}\right)$$
(Eq. 5)

$$V_2(\theta_2)=\frac{\hbar^2}{2\mu}\left(\frac{\upsilon_2(\upsilon_2+1)}{\sin^2\theta_2}-2\rho_2\cot\theta_2\right)$$
(Eq. 6)

$$V_3(\theta_3)=\frac{\hbar^2}{2\mu}\left(\frac{\upsilon_3(\upsilon_3+1)}{\sin^2\theta_2}-2\rho_3\cot\theta_3\right)$$
(Eq. 7)

$$V_4(\theta_4)=\frac{\hbar^2}{2\mu}\left(\frac{\upsilon_4(\upsilon_4+1)}{\sin^2\theta_4}-2\rho_4\cot\theta_4\right)$$
(Eq. 8)

$$V_5(\theta_5)=\frac{\hbar^2}{2\mu}\left(\frac{\upsilon_5(\upsilon_5+1)}{\sin^2\theta_5}-2\rho_5\cot\theta_5\right)$$
(Eq. 9)

Where equation (3) is a special form of equation (25) for the 6-dimensional system. With $0\le\theta_1\le 2\pi$, and $0\le\theta_2,\theta_3,\theta_4,\theta_5\le\pi$. $\hbar$, μ, and ω are Planck constant, mass, and frequency of a particle. α, σ, υ_i, and ρ_i are the potential

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parameters, respectively.

The thermodynamics properties for quantum mechanical systems are an interesting problem to consider because of their interactions with the energy spectrum that contains the physical properties of the quantum system its self. There are many studies of thermodynamic properties. Some of them are the thermodynamical properties for systems with double ring-shaped quantum dot type potential (Khordad and Sedehi, 2018), thermodynamical properties from Klein Gordon equation with DFPEP potential D-dimensional spaces (A N Ikot *et al*., 2016). For example, thermodynamical properties from Schrodinger Equation and Klein Gordon equation with Poshcl-Teller potential (Yahya and Oyewumi, 2016), thermodynamical properties of Schrodinger equation with an anharmonic oscillator in cosmic string framework (Sobhani, *et al*., 2018), thermodynamical properties of diatomic molecules using general molecular potential (A N Ikot *et al*., 2018).

The vibrational mean energy U, the vibrational specific heat C, the vibrational free energy F, and the vibrational entropy S can be solved using some following order. Starting by solving the Shrodinger equation in the D-dimensional system to get the energy level equations and eigenfunctions for the angular and radial parts, respectively. The energy equation is then employed to determine the partition function, which is the principal element for defining the thermodynamics properties.

The research of thermodynamics properties in a D-dimensional system with ring shape type potentials is still an interesting topic in the quantum mechanics. Generally, ring-shaped type potential in D-dimensional space is analyzed without being extended to study thermodynamic properties. The complexity of the theory and application in quantum mechanical systems analyzed the thermodynamics properties of the Schrodinger equation in 6-dimensional spaces with DRSO plus Manning Rosen potential using the SUSY QM technique.

This study aimed to find the nonrelativistic energy spectra equation, radial wave function and angular wave functions, and also the corresponding thermodynamics properties such us the vibrational mean energy, vibrational free energy, vibrational specific heat and vibrational entropy of six-dimensional Schrodinger equation for Double Ring-Shape Oscillator (DRSO) and Manning Rosen potential.

2. MATERIALS AND METHODS:

2.1. Supersymmetry Quantum Mechanic (SUSY QM) Method

One of the powerful techniques to solve Schrodinger-like equations is Supersymmetry Quantum Mechanics (SUSY QM) method. The method proposed by Witten is principally developed based on the existence of fermionic operators which are commute with the Hamiltonian (Witten, 1981) Concerning the supersymmetry system in general, the Hamiltonian H is composed of the square of supersymmetry charges and can be expressed as a multiplication between a pair of supersymmetry operators as

$$H_{\mp}(x) = A^{\pm}A^{\mp} \quad \text{(Eq. 10)}$$

with

$$A^{\pm} = \mp\frac{\hbar}{\sqrt{2\mu}}\frac{d}{dx} + W(x) \quad \text{(Eq. 11)}$$

where A^{+} and A^{-} are rising and lowering operators, and a pair of SUSY QM partner potentials $V_{\mp}(x)$ gave as

$$V_{\mp}(x) = W^{2}(x) \mp \frac{\hbar}{\sqrt{2\mu}}\frac{dW(x)}{dx} \quad \text{(Eq. 12)}$$

The general Hamiltonian can be factored into

$$H = H_{-} + E_0 = -\frac{\hbar^2}{2\mu}\frac{d^2}{dx^2} + V_{-}(x;a_0) + E_0 \quad \text{(Eq. 13)}$$

Therefore, using equations (12) and (13) it has got the equation 14

$$V(x) = V_{-}(x;a_0) + E_0 = W^{2}(x) - \frac{\hbar}{\sqrt{2\mu}}\frac{dW(x)}{dx} + E_0 \quad \text{(Eq. 14)}$$

where $V(x)$ is an effective potential and E_0 is the groundstate energy. Superpotential $W(x)$ is hypothetically determined by considering the effective potential equation form of the related system expressed in equation (14).

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The supersymmetry solely presents the relationship between the eigenvalues and eigenfunctions between two Hamiltonian partners but does not provide the actual spectrum Consequently, one has to consider shape invariance condition. The potential is said to shape invariance if its supersymmetry partner potential has similar shapes but not for the parameters. More specifically, if $V_-(x;a_j)$ is a potential with its partner superpotential $V_+(x;a_j)$ have to satisfy the equation

$$V_+(x;a_j) = V_-(x;a_{j+1}) + R(a_{j+1})$$
(Eq. 15)

with

$$V_+(x;a_j) = W^2(x;a_j) + \frac{\hbar}{\sqrt{2m}}\frac{dW(x;a_j)}{dx} \ ;$$
$$V_-(x;a_{j+1}) = W^2(x;a_{j+1}) - \frac{\hbar}{\sqrt{2m}}\frac{dW(x;a_{j+1})}{dx}$$
(Eq. 16)

where $j = 0,1,2,\ldots$ and a is a parameter of the potential V_- when its ground state energy is zero, $a_j = f_j(a_0)$ for f_j is a function that is applied j times, $R(a_j)$ is an independent Constanta towards x. Furthermore, the Hamiltonian eigenvalue can be written as (Dutt, *et al.*, 1988; Khare and Bhaduri, 1993).

$$E_n{}^{(-)} = \sum_{k=1}^{n} R(a_k)$$
(Eq. 17)

Thus, from equation (13) and (17), it is possible to get the energy spectrum of the system

$$E_n = E_n{}^{(-)} + E_0$$
(Eq. 18)

Ground state wave function of H_- which the energy is zero could be obtained by

$$A^-\psi_0{}^{(-)} = 0$$
(Eq. 19)

The excited wave function $\psi_1{}^{(-)}(x;a_0),\cdots,\psi_n^{(-)}(x;a_0)$ from H_- can be determined by applying raising operator on the lower wave function

$$\psi_n{}^{(-)}(x;a_0) \approx A^+(x;a_0)A^+(x;a_1)\cdots A^+(x;a_{n-1}),\psi_0^{(-)}(x;a_n)$$
(Eq. 20)

2.2. Schrodinger Equation in 6-Dimensional Coordinates

The time-independent Schrodinger equation with potential in D-dimensional is given by (Hassanabadi, *et al.*, 2011; Wang, *et al.*, 2002)

$$\left(\nabla_D^2 - \frac{2\mu}{\hbar^2}V\right)\psi_{l_1,\ldots,l_{D-1}}^{(l_{D-1}=l)}(\hat{X}) = -\frac{2\mu}{\hbar^2}E\psi_{l_1,\ldots,l_{D-1}}^{(l_{D-1}=l)}(\hat{X})$$
(Eq. 21)

with E is the energy of the particle. $\psi_{l_1,\ldots,l_{D-1}}^{(l_{D-1}=l)}(\hat{X})$ is the Laplace operator for D-dimensional space, ∇_D^2 and D-dimensional position vector are given as

$$\nabla_D^2 = \frac{1}{h}\sum_{j=0}^{D-1}\frac{d}{d\theta_j}\left(\frac{h}{h_j^2}\frac{\partial}{\partial\theta_j}\right)$$
(Eq. 22)

$$\theta_0 = r; h = \prod_{j=0}^{D-1} h_j; h_j^2 = \sum_{i=0}^{D}\left(\frac{\partial X_i}{\partial\theta_j}\right)^2$$
(Eq. 23)

The relevance between D-dimensional position vector $\hat{X} = (r,\theta_i)$, and hyperspherical Cartesian coordinates x_1 can be written as follows:

$$x_1 = r\cos\theta_1\sin\theta_2,\cdots,\sin\theta_{D-1}$$
$$x_2 = r\sin\theta_1\sin\theta_2,\cdots,\sin\theta_{D-1}$$
$$x_b = r\cos\theta_{b-1}\sin\theta_b,\cdots,\sin\theta_{D-1}$$
$$\vdots$$
$$x_{D-1} = r\cos\theta_{D-2}\sin\theta_{D-1}$$
$$x_D = r\cos\theta_{D-1}$$
(Eq. 24)

where $r \in [0,\infty]$, $\theta_1 \in [0,2\pi]$, $\theta_b \in [0,\pi]$, $D = 2,3,\cdots$, and $b = 3,4,\cdots,(D-1)$ (Dong, 2011). Specifically, for the case when $D = 3$, the hyperspherical coordinate system reduced into spherical coordinates (r, θ, φ).

The separable D dimensional non-central potential hypothetically proposed as

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$$V(r,\theta_1,\theta_2,\dots,\theta_{D-1}) = V(r) + \frac{1}{r^2}\sum_{j=1}^{D-1}\frac{V_j(\theta_j)}{\sin^2\theta_{j+1}\dots\sin^2\theta_{D-1}} + \dots + \frac{V_{D-2}(\theta_{D-2})}{\sin^2\theta_{D-1}} + V_{D-1}(\theta_{D-1})$$

(Eq. 25)

For 6 dimensional quantum system with a separable 6 dimensional non central potential it has got the D dimensional coordinate system obtained from equation (24) as

$$\begin{aligned} x_1 &= r\cos\theta_1\sin\theta_2\sin\theta_3\sin\theta_4\sin\theta_5 \\ x_2 &= r\sin\theta_1\sin\theta_2\sin\theta_3\sin\theta_4\sin\theta_5 \\ x_3 &= r\cos\theta_2\sin\theta_3\sin\theta_4\sin\theta_5 \\ x_4 &= r\cos\theta_3\sin\theta_4\sin\theta_5 \\ x_5 &= r\cos\theta_4\sin\theta_5 \\ x_6 &= r\cos\theta_5 \end{aligned}$$

(Eq 26)

with the coordinate scale obtained from equation (23) as

$$\begin{aligned} h_1 &= r\sin\theta_2\sin\theta_3\sin\theta_4\sin\theta_5 \\ h_2 &= r\sin\theta_3\sin\theta_4\sin\theta_5 \\ h_3 &= r\sin\theta_4\sin\theta_5 \\ h_4 &= r\sin\theta_5 \\ h_5 &= r \\ h_0 &= h_r = 1 \end{aligned}$$

(Eq. 27)

Apply equations (26) and (27) on equation (22), it has got the Laplacian operator for the 6-Dimensional system as

$$\nabla_6^2 = \frac{1}{r^5}\frac{\partial}{\partial r}\left(r^5\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\left\{\frac{1}{\sin^2\theta_2\sin^2\theta_3\sin^2\theta_4\sin^2\theta_5}\left(\frac{\partial^2}{\partial\theta_1^2}\right) + \frac{1}{\sin^2\theta_3\sin^2\theta_4\sin^2\theta_5}\left[\frac{1}{\sin\theta_2}\frac{\partial}{\partial\theta_2}\left(\sin\theta_2\frac{\partial}{\partial\theta_2}\right)\right]\right\} + \frac{1}{r^2}\left\{\frac{1}{\sin^2\theta_4\sin^2\theta_5}\left[\frac{1}{\sin^2\theta_3}\frac{\partial}{\partial\theta_3}\left(\sin^2\theta_3\frac{\partial}{\partial\theta_3}\right)\right] + \frac{1}{\sin^2\theta_5}\left[\frac{1}{\sin^3\theta_4}\frac{\partial}{\partial\theta_4}\left(sin^3\theta_4\frac{\partial}{\partial\theta_4}\right)\right] + \frac{1}{\sin^4\theta_5}\frac{\partial}{\partial\theta_5}\left(\sin^4\theta_5\frac{\partial}{\partial\theta_5}\right)\right\}$$

(Eq. 28)

The potential form that allows the variable separation in 6-dimensional system obtained from equation (25) have been given in equations (3-9).

By setting up

$$\psi_{l_1,\dots,l_{D-1}}^{(l_{D-1}=l)}(\hat{X}) = R(r)P_i(\theta_i) \;;\; i = 1,2,3,4,5$$

(Eq. 29)

then it is possible to make the separation of variables by applying equations (27) - (29) into (21). Using simple mathematical procedures, it can be obtained wave equations of the 6-dimensional system for radial r and angular θ_i functions, respectively.

$$r^2\frac{1}{r^5}\frac{d}{dr}\left(r^5\frac{dR(r)}{dr}\right) + \frac{2\mu}{\hbar^2}(Er^2 - V_1(r)r^2)R(r) - \lambda_5 R(r) = 0$$

(Eq. 30)

$$\left\{\frac{d^2}{d{\theta_1}^2} - \frac{2\mu}{\hbar^2}V_1(\theta_1) + \lambda_1\right\}P_1 = 0$$

(Eq. 31)

$$\frac{1}{\sin\theta_2}\frac{d}{d\theta_2}\left(\sin\theta_2\frac{dP_2}{d\theta_2}\right) - \left\{\frac{2\mu}{\hbar^2}V_2(\theta_2) + \frac{\lambda_1}{\sin^2\theta_2} - \lambda_2\right\}P_2 = 0$$

(Eq. 32)

$$\frac{1}{\sin^2\theta_3}\frac{d}{d\theta_3}\left(\sin^2\theta_3\frac{dP_3}{d\theta_3}\right) - \left\{\frac{2\mu}{\hbar^2}V_3(\theta_3) + \frac{\lambda_2}{\sin^2\theta_3} - \lambda_3\right\}P_3 = 0$$

(Eq. 33)

$$\frac{1}{\sin^3\theta_4}\frac{d}{d\theta_4}\left(\sin^3\theta_4\frac{dP_4}{d\theta_4}\right) - \left\{\frac{2\mu}{\hbar^2}V_4(\theta_4) + \frac{\lambda_3}{\sin^2\theta_4} - \lambda_4\right\}P_4 = 0$$

(Eq. 34)

$$\frac{1}{\sin^4\theta_5}\frac{d}{d\theta_5}\left(\sin^4\theta_5\frac{dP_5}{d\theta_5}\right) - \left\{\frac{2\mu}{\hbar^2}V_5(\theta_5) + \frac{\lambda_4}{\sin^2\theta_5} - \lambda_5\right\}P_5 = 0$$

(Eq. 35)

with λ_1 - λ_5 are the variable separation constants.

Each of these equations (30), (32) – (35) has both first and second derivative forms, so we need to reduce these equations to the form of a Schrodinger-like equation with the second derivative form only. Therefore, we set $R(r) = r^{-\left(\frac{D-1}{2}\right)}\chi(r)$ and $P_i = \sin^{-\left(\frac{i-1}{2}\right)}\theta_i\, Q(\theta_i)$, *i = 2-5* then using simple mathematic operations, it can be obtained

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$$\left\{\frac{d^2}{dr^2}-\frac{2\mu}{\hbar^2}V_r(r)-\frac{\lambda_5+\frac{15}{4}}{r^2}+\frac{2\mu}{\hbar^2}E\right\}\chi(r)=0$$
(Eq. 36)

$$\left\{\frac{d^2}{d{\theta_1}^2}-\frac{2\mu}{\hbar^2}V_1(\theta_1)+\lambda_1\right\}Q_1(\theta_1)=0$$
(Eq. 37)

$$\left\{\frac{d^2}{d{\theta_2}^2}-\frac{2\mu}{\hbar^2}V_2(\theta_2)-\frac{\lambda_1-\frac{1}{4}}{\sin^2\theta_2}+\left(\frac{1}{4}+\lambda_2\right)\right\}Q_2(\theta_2)=0$$
(Eq. 38)

$$\left\{\frac{d^2}{d{\theta_3}^2}-\frac{2\mu}{\hbar^2}V_3(\theta_3)-\frac{\lambda_2}{\sin^2\theta_3}+(1+\lambda_3)\right\}Q_3(\theta_3)=0$$
(Eq. 39)

$$\left\{\frac{d^2}{d{\theta_4}^2}-\frac{2\mu}{\hbar^2}V_4(\theta_4)-\frac{\lambda_3+\frac{3}{4}}{\sin^2\theta_4}+\left(\frac{9}{4}+\lambda_4\right)\right\}Q_4(\theta_4)=0$$
(Eq. 40)

$$\left\{\frac{d^2}{d{\theta_5}^2}-\frac{2\mu}{\hbar^2}V_5(\theta_5)-\frac{\lambda_4+2}{\sin^2\theta_5}+(4+\lambda_5)\right\}Q_5(\theta_5)=0$$
(Eq. 41)

the radial Schrodinger-like wave equation (36) and angular Schrodinger-like wave equations (37) – (41) are solvable using the SUSY QM method.

3. RESULTS AND DISCUSSION:

3.1. The solution of the Schrodinger Equation in 6-Dimensional System

3.1.1 Solution of angular $\boldsymbol{\theta_1}$

Substitute the angular θ_1 potential in equations (3) and (5) into equation (37)

$$\left\{-\frac{\hbar^2}{2\mu}\frac{d^2}{d{\theta_1}^2}+\frac{\hbar^2}{2\mu}\left(\frac{\alpha}{\sin^2\theta_1}+\frac{\sigma}{\cos^2\theta_1}\right)\right\}Q_1(\theta_1)=\frac{\hbar^2}{2\mu}\lambda_1 Q_1(\theta_1)$$
(Eq. 42)

then it has got its effective potential as,

$$V_{\mathrm{eff}}(\theta_1)=\frac{\hbar^2}{2\mu}\left(\frac{\alpha'(\alpha'-1)}{\sin^2\theta_1}+\frac{\sigma'(\sigma'-1)}{\cos^2\theta_1}\right)$$
(Eq. 43)

where $\alpha'=\sqrt{\alpha+\frac{1}{4}}+\frac{1}{2}$, $\sigma'=\sqrt{\sigma+\frac{1}{4}}+\frac{1}{2}$ and $E_{\theta_1}=\frac{\hbar^2}{2\mu}\lambda_1$.

Using the following hypothetical superpotential for the effective potential in equation (43),

$$W(\theta_1)=\frac{\hbar}{\sqrt{2\mu}}(M\cot\theta_1+N\tan\theta_1)$$
(Eq. 44)

then inserting equations (43) and (44) into equation (14) it is possible to obtain

$$\frac{\hbar^2}{2\mu}\left(\frac{\alpha'(\alpha'-1)}{\sin^2\theta_1}+\frac{\sigma'(\sigma'-1)}{\cos^2\theta_1}\right)-E_0=\frac{\hbar^2}{2\mu}\left(\frac{M^2+M}{\sin^2\theta_1}+\frac{N^2-N}{\cos^2\theta_1}\right)-\frac{\hbar^2}{2\mu}(M-N)^2$$
(Eq. 45)

It has got several parameter relations as follows

$$M(M+1)=\alpha'(\alpha'-1)\rightarrow M=-\alpha'$$
(Eq. 46)

$$N(N-1)=\sigma'(\sigma'-1)\rightarrow N=\sigma'$$
(Eq. 47)

$$E_0(\theta_1)=(M-N)^2=(\alpha'+\sigma')^2$$
(Eq. 48)

The superpartner potential for angular θ_1 is obtained by substituting equation (44) and (46) - (48) into (12)

$$V_-(\theta_1;a_0)=\frac{\hbar^2}{2\mu}\left(\frac{\alpha'(\alpha'-1)}{\sin^2\theta_1}+\frac{\sigma'(\sigma'-1)}{\cos^2\theta_1}\right)-\frac{\hbar^2}{2\mu}(\alpha'+\sigma')^2$$
(Eq. 49)

$$V_+(\theta_1;a_0)=\frac{\hbar^2}{2\mu}\left(\frac{\alpha'(\alpha'+1)}{\sin^2\theta_1}+\frac{\sigma'(\sigma'+1)}{\cos^2\theta_1}\right)-\frac{\hbar^2}{2\mu}(\alpha'+\sigma')^2$$
(Eq. 50)

These superpartner potential are similar and differ only in their parameters. Consequently, to obtain $V_{\mp}(\theta_1;a_{1,2,\ldots},a_n)$ it is necessary to shift these parameters $\alpha'\rightarrow\alpha'+1$ and $\sigma'\rightarrow\sigma'+1$.

By generalization using a characteristic of shape invariance (15), it is obtained

$$R(\theta_1;a_n)=-\frac{\hbar^2}{2\mu}((\alpha'+\sigma'+2(n-1))^2+(\alpha'+\sigma'+2n)^2)$$
(Eq. 51)

Therefore, by using equation (17) and (18), the eigenvalue for angular part θ_1 can be easily obtained as

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$$E_{n_l}(\theta_1) = \frac{\hbar^2}{2\mu}(\alpha' + \sigma' + 2n_{l1})^2$$

(Eq. 52)

hence, it has got

$$\lambda_1 = (\alpha' + \sigma' + 2n_{l1})^2$$

(Eq. 53)

where n_l is the orbital quantum number.

Applying the superpotential (44) on equation (11) has raised and lowered operator, respectively.

$$A^+(\theta_1) = -\frac{\hbar}{\sqrt{2\mu}}\frac{d}{d\theta_1} + \frac{\hbar}{\sqrt{2\mu}}(-\alpha'\cot\theta_1 + \sigma'\tan\theta_1)$$

(Eq. 54)

$$A^-(\theta_1) = \frac{\hbar}{\sqrt{2\mu}}\frac{d}{d\theta_1} + \frac{\hbar}{\sqrt{2\mu}}(-\alpha'\cot\theta_1 + \sigma'\tan\theta_1)$$

(Eq. 55)

Thus, using equation (55) and (19) it is obtained the ground-state angular θ_1 wave function as

$$\psi_0{}^{(-)}(\theta_1) = \aleph_{\theta_1}(\sin\theta_1)^{\alpha'}(\cos\theta_1)^{\sigma'}$$

(Eq. 56)

with $\aleph_{\theta_1}$ is the normalization constant of angular θ_1 wave function.

3.1.2 Solution of angular $\boldsymbol{\theta_2}$

Substitute the angular θ_1 potential in equations (3) and (6) into equation (38)

$$-\frac{\hbar^2}{2\mu}\frac{d^2Q_2(\theta_2)}{d\theta_2{}^2} + \frac{\hbar^2}{2\mu}\left(\frac{\upsilon_2(\upsilon_2+1) + \left(\lambda_1 - \frac{1}{4}\right)}{\sin^2\theta_2} - 2\rho_2\cot\theta_2\right)Q_2(\theta_2) = \frac{\hbar^2}{2\mu}\left(\frac{1}{4} + \lambda_2\right)Q_2(\theta_2)$$

(Eq. 57)

thus it is obtained its effective potential as,

$$V_{\text{eff}}(\theta_2) = \frac{\hbar^2}{2\mu}\left(\frac{\upsilon_2'(\upsilon_2'+1)}{\sin^2\theta_2} - 2\rho_2\cot\theta_2\right)$$

(Eq. 58)

where $\upsilon_2' = \sqrt{\upsilon_2(\upsilon_2+1) + \lambda_1} - \frac{1}{2}$ and $E_{\theta_2} = \frac{\hbar^2}{2\mu}\left(\frac{1}{4} + \lambda_2\right)$.

By using following hypothetical superpotential for the effective potential (58),

$$W(\theta_2) = \frac{\hbar}{\sqrt{2\mu}}\left(B_2\cot\theta_2 + \frac{C}{B_2}\right)$$

(Eq. 59)

And inserting (58) and (59) into equation (14)

$$\frac{\hbar^2}{2\mu}\left(\frac{\upsilon_2'(\upsilon_2'+1)}{\sin^2\theta_2} - 2\rho_2\cot\theta_2\right) - E_0(\theta_2) = \frac{\hbar^2}{2\mu}\left(\frac{B_2{}^2 + B_2}{\sin^2\theta_1} - 2\rho_2\cot\theta_2\right) + \frac{\hbar^2}{2\mu}\left(\frac{C^2}{B_2} - B_2{}^2\right)$$

(Eq. 60)

It is got several parameter relations as follows

$$B_2 = \upsilon_2'$$

(Eq. 61)

$$C_2 = \rho_2$$

(Eq. 62)

$$E_0(\theta_2) = -\frac{\hbar^2}{2\mu}\left(\frac{C^2}{B_2} - B_2{}^2\right)$$

(Eq. 63)

The superpartner potential for the angular θ_2 part is obtained by substituting equation (59) and (61) – (63) into (12).

$$V_-(\theta_2; a_0) = \frac{\hbar^2}{2\mu}\left(\frac{\upsilon_2'(\upsilon_2'+1)}{\sin^2\theta_2} - 2\rho_2\cot\theta_2\right) + \frac{\hbar^2}{2\mu}\left(\frac{C^2}{\upsilon_2'} - \upsilon_2'{}^2\right)$$

(Eq. 64)

$$V_+(\theta_2; a_0) = \frac{\hbar^2}{2\mu}\left(\frac{\upsilon_2'(\upsilon_2'-1)}{\sin^2\theta_2} - 2\rho_2\cot\theta_2\right) + \frac{\hbar^2}{2\mu}\left(\frac{C^2}{\upsilon_2'} - \upsilon_2'{}^2\right)$$

(Eq. 65)

These superpartner potential (64) and (66) are similar and differ only on their parameters. Consequently, to obtain $V_{\mp}(\theta_2; a_{1,2,\dots}, a_n)$ it is necessary to shift these parameters $\upsilon' \to \upsilon' - 1$.

By generalization using a characteristic of shape invariance (15), it is obtained

$$R(\theta_2; a_n) = \frac{\hbar^2}{2\mu}\left(\frac{\rho_2^2}{\upsilon_2'{}^2} - \upsilon_2'{}^2\right) - \frac{\hbar^2}{2\mu}\left(\frac{\rho_2^2}{(\upsilon_2' - n)^2} - (\upsilon_2' - n)^2\right)$$

(Eq. 66)

Therefore, by using equation (17), (18) and (63), the eigenvalue for angular θ_2 can be obtained easily as

$$E_{n_l}(\theta_2) = -\frac{\hbar^2}{2\mu}\left(\frac{\rho_2^2}{(\upsilon_2' - n_{l2})^2} - (\upsilon_2' - n_{l2})^2\right)$$

(Eq. 67)

hence, it is got

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$$\lambda_2 = -\frac{\rho_2^2}{\left(\sqrt{\upsilon_2(\upsilon_2+1)+\lambda_1}-n_{l2}\right)^2} + \left(\sqrt{\upsilon_2(\upsilon_2+1)+\lambda_1}-n_{l2}\right)^2 - \frac{1}{4}$$
(Eq. 68)

Apply the superpotential (59) on equation (11), then it has got raising and lowering operator, respectively.

$$A^{+}(\theta_2; a_0) = -\frac{\hbar}{\sqrt{2\mu}}\frac{d}{d\theta_2} + \frac{\hbar}{\sqrt{2\mu}}\left(\upsilon'_2 \tan\theta_2 + \frac{\rho_2}{\upsilon'_2}\right)$$
(Eq. 69)

$$A^{-}(\theta_2; a_0) = \frac{\hbar}{\sqrt{2\mu}}\frac{d}{d\theta_2} + \frac{\hbar}{\sqrt{2\mu}}\left(\upsilon'_2 \tan\theta_2 + \frac{\rho_2}{\upsilon'_2}\right)$$
(Eq. 70)

Thus, using equation (70) and (19) it is obtained the ground-state angular θ_2 wave function as

$$P_0{}^{(-)}(\theta_2) = \aleph_{\theta_2}(\sin\theta_2)^{\upsilon'_2} e^{-\frac{\rho_2}{\upsilon'_2}\theta_2}$$
(Eq. 71)

with $\aleph_{\theta_2}$ is the normalization constant of angular θ_2 wave function.

3.1.3 Solution of angular θ_3, θ_4, and θ_5

Using similar steps as well as the solution for θ_2 it is obtained the λ_3, λ_4, and λ_5, respectively.

$$\lambda_3 = \left(-\frac{\rho_3^2}{(\upsilon'_3 - n_{l3})^2} + (\upsilon'_3 - n_{l3})^2\right) - 1$$
(Eq. 72)

$$\lambda_4 = \left(-\frac{\rho_4^2}{(\upsilon'_4 - n_{l4})^2} + (\upsilon'_4 - n_{l4})^2\right) - \frac{9}{4}$$
(Eq. 73)

$$\lambda_5 = \left(-\frac{\rho_5^2}{(\upsilon'_5 - n_{l5})^2} + (\upsilon'_5 - n_{l5})^2\right) - 4$$
(Eq. 74)

where,

$$\upsilon'_3 = \sqrt{\upsilon_3(\upsilon_3+1)+\lambda_2+\frac{1}{4}} - \frac{1}{2}$$
(Eq. 75)

$$\upsilon'_4 = \sqrt{\upsilon_4(\upsilon_4+1)+\lambda_3+1} - \frac{1}{2}$$
(Eq. 76)

$$\upsilon'_5 = \sqrt{\upsilon_3(\upsilon_3+1)+\lambda_4+\frac{9}{4}} - \frac{1}{2}$$
(Eq. 77)

Furthermore, the groundstate wave function for angular θ_3, θ_4, and θ_5 are

$$P_0{}^{(-)}(\theta_3) = \aleph_{\theta_3}(\sin\theta_3)^{\upsilon'_3} e^{-\frac{\rho_3}{\upsilon'_3}\theta_3}$$
(Eq. 78)

$$P_0{}^{(-)}(\theta_4) = \aleph_{\theta_4}(\sin\theta_4)^{\upsilon'_4} e^{-\frac{\rho_4}{\upsilon'_4}\theta_4}$$
(Eq. 79)

$$P_0{}^{(-)}(\theta_5) = \aleph_{\theta_5}(\sin\theta_5)^{\upsilon'_5} e^{-\frac{\rho_5}{\upsilon'_5}\theta_5}$$
(Eq. 80)

with $\aleph_{\theta_3}$, $\aleph_{\theta_4}$, and $\aleph_{\theta_5}$ are the normalization constant.

3.1.4 Solution of radial part

The radial Schrodinger equation (36) with DRSO plus Manning Rosen potential (9) in D-dimensional space

$$\left\{-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2} + \frac{1}{2}\mu\omega^2 r^2 + \frac{\hbar^2}{2\mu}\frac{\lambda_5 + \frac{15}{4}}{r^2} - E\right\}\chi(r) = 0$$
(Eq. 81)

has the following effective potential

$$V_{\text{eff}}(r) = \frac{1}{2}\mu\omega^2 r^2 + \frac{\hbar^2}{2\mu}\frac{m(m+1)}{r^2}$$
(Eq. 82)

where $m(m+1) = \lambda_5 + \frac{15}{4}$, or

$$m = \sqrt{\lambda_5 + \frac{15}{4} + \frac{1}{4}} - \frac{1}{2}$$
(Eq. 83)

Further, substituting this radial effective potential (82) with its hypothetical superpotential (84)

$$W(r) = Kr + \frac{V}{r}$$
(Eq. 84)

into equation (14)

$$\frac{1}{2}\mu\omega^2 r^2 + \frac{\hbar^2}{2\mu}\frac{m(m+1)}{r^2} - E_0(r) = \left(K^2 r^2 + \frac{V(V+1)}{r^2}\right) + 2K\left(V - \frac{1}{2}\right)$$
(Eq. 85)

then it has got some following relation

$$K = \omega\sqrt{\frac{\mu}{2}}$$
(Eq. 86)

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$$V = -\frac{\hbar}{\sqrt{2\mu}}(m+1)$$
(Eq. 87)

$$E_0(r) = \omega\hbar\left(m + \frac{3}{2}\right)$$
(Eq. 88)

As well as previous work on the angular part, now it is possible to determine the superpotential of the radial part by applying equations (44), (46) – (48) into equation (12).

$$V_{-}(r;a_0) = \frac{\mu\omega^2}{2}r^2 + \frac{\hbar^2}{2\mu}\frac{m(m+1)}{r^2} - \omega\hbar\left(m + \frac{3}{2}\right)$$
(Eq. 89)

$$V_{+}(r;a_0) = \frac{\mu\omega^2}{2}r^2 + \frac{\hbar^2}{2\mu}\frac{(m+1)(m+2)}{r^2} - \omega\hbar\left(m + \frac{1}{2}\right)$$
(Eq. 90)

Via a mapping parameter $m \rightarrow m+1$ on equation (89) and (90), it has got $V_{\mp}(r;a_1,a_2,\ldots,a_n)$. Further, using the characteristic of shape invariance (15) and (17) it is obtained

$$R(r;a_n) = 2\omega\hbar$$
(Eq. 91)

and $E_n{}^{(-)} = 2n\omega\hbar$. Therefore, using equation (18) it has got the energy eigenvalue for the radial part as

$$E_{n_r}(r) = \omega\hbar\left(\left(\sqrt{\lambda_5 + \frac{15}{4} + \frac{1}{4}} - \frac{1}{2}\right) + \frac{3}{2} + 2n_r\right)$$
(Eq. 92)

The raising and lowering operators are obtained using superpotential $\mathrm{W}(r)$ (84) and equation (11)

$$A^{+}(r) = -\frac{\hbar}{\sqrt{2\mu}}\frac{d}{dr} + \sqrt{\frac{\mu}{2}}\omega r - \frac{\hbar}{\sqrt{2\mu}}\frac{(m+1)}{r}$$
(Eq. 93)

$$A^{-}(r) = \frac{\hbar}{\sqrt{2\mu}}\frac{d}{dr} + \sqrt{\frac{\mu}{2}}\omega r - \frac{\hbar}{\sqrt{2\mu}}\frac{(m+1)}{r}$$
(Eq. 94)

Finally, applying equation (94) on (19) and continue with equation (93) and (20) it is obtained the ground-state and first excited radial r wave function, respectively,

$$R_0{}^{(-)}(r) = \aleph_r r^{\left(\left(\sqrt{\lambda_5 - \frac{15}{4} + \frac{1}{4}} - \frac{1}{2}\right) + 1\right)} e^{-\left(\frac{\mu}{2\hbar}\omega r^2\right)}$$
(Eq. 95)

$$R_1{}^{(-)}(r) = \aleph_r\left(\sqrt{2\mu}\omega r - \frac{\hbar}{\sqrt{2\mu}}\frac{2\left(m + \frac{3}{2}\right)}{r}\right)\left(r^{(m+2)}\right)e^{-\left(\frac{\mu}{2\hbar}\omega r^2\right)}$$
(Eq. 96)

with $\aleph_r$ is the normalization constant of the radial r wave function. Since the $\int_{-\infty}^{\infty}|\aleph_x\psi_0|^2 dx = 1$, then it has got

$$\aleph_r = \sqrt{\frac{\left(\frac{\mu\omega}{2\hbar}\right)^{\left(\frac{2m+3}{2}\right)}}{\Gamma\left(\frac{2m+3}{2}\right)}}$$
(Eq. 97)

3.2. Thermodynamics Properties

To obtain some thermodynamics properties such as the vibrational mean energy U, specific heat C, vibrational free energy F, and vibrational entropy S it is necessary to derive the partition function Z equation first. This function is also known as the distribution function and its existence is important since someone wants to study the thermodynamical properties. Furthermore, this function can be derived using the energy equation from the considered system (Akpan N Ikot et al., 2018; Suparmi, Cari, and Pratiwi, 2016). The partition function for the 6-dimensional system with DRSO plus Manning Rosen potentials can be written as

$$Z(\zeta,\beta) = \sum_{n=0}^{\zeta} e^{-\beta E_n}, \beta = \frac{1}{k_B T}$$
(Eq. 98)

with k_B and T are Boltzmann constant and temperature, E_n is the nonrelativistic energy from the system. Rewrite the energy equation (92)

$$E_{nr} = \omega\hbar\left(m + \frac{1}{2} + 2n_r + 1\right)$$
(Eq. 99)

with $m = \lambda_5 + \frac{15}{4} + \frac{1}{4} - \frac{1}{2}$.

Therefore it is obtained the vibrational partition function Z as follow

$$Z(\beta) = \sum_{n=0}^{\infty} e^{-\beta\omega\hbar\left(m + \frac{1}{2} + 2n_r + 1\right)} = e^{-\beta\omega\hbar\left(m + \frac{1}{2}\right)}\frac{1}{2\sinh(\beta\omega\hbar)}$$
(Eq. 100)

3.2.1 Vibrational Mean Energy

The vibrational mean energy U for DRSO

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plus Manning Rosen potential in 6-dimensional space is

$$U(\beta) = -\frac{\partial}{\partial\beta} ln(Z)$$
$$= -\frac{\partial}{\partial\beta} \ln\left(e^{-\beta\omega\hbar\left(m+\frac{1}{2}\right)} \frac{1}{2\sinh(\beta\omega\hbar)}\right)$$
$$= \omega\hbar\left(m+\frac{1}{2}\right) + \omega\hbar\coth(\beta\omega\hbar)$$

(Eq. 101)

3.2.2 Vibrational Specific Heat

The vibrational mean energy C for DRSO plus Manning Rosen potential in 6-dimensional space is

$$C(\beta) = -\frac{\partial U}{\partial T} = -k_B\beta^2\left(\frac{\partial U}{\partial\beta}\right)$$
$$= k_B(\omega\hbar\beta)^2 \mathrm{csch}^2(\beta\omega\hbar)$$

(Eq. 102)

3.2.3 Vibrational Free Energy

The vibrational free energy F for the DRSO plus Manning Rosen potential in D-dimensional system is solved as follows

$$F = -\frac{1}{\beta}\mathrm{ln}Z = \omega\hbar\left(m+\frac{1}{2}\right) + \frac{\ln\left(2\sinh(\beta\omega\hbar)\right)}{\beta}$$

(Eq. 103)

3.2.4 Vibrational Entropy

The vibrational entropy S for the DRSO plus Manning Rosen potential in D-dimensional system is solved as follows

$$S(\beta) = k\mathrm{ln}Z + kT\left(\frac{\partial \mathrm{ln}Z}{\partial T}\right)$$
$$S(\beta) = k\beta\omega\hbar\coth(\beta\omega\hbar) - k\mathrm{ln}\left(2\sinh(\beta\omega\hbar)\right)$$

(Eq. 104)

3.3. Discussion

The analysis of the energy values, wave functions, and thermodynamics properties are explained by graphical representation. To obtain the graphics of energy versus radial quantum number n_r, equation (92) was used by substituting the separable constants λ_5 (74), λ_4 (73), λ_3 (72), λ_2 (68), and λ_1 (53).

From Figure 1, the energy versus radial quantum number n_r was plotted for various orbital quantum number n_l by set the $\hbar = \mu = 1, \omega = \alpha = \sigma = v_i = \rho_i = 5$. The graphic shows that every increment of the orbital quantum number tends to decrease energy, with the increment of n_{l5} causes more decrement than other orbital quantum numbers. Exceptionally for the n_{l1}, when its increase caused a decrease at energy. From Figure 2, the energy versus radial quantum number n_r was plotted for various values of the potential parameters. It shows that in general, the increment of potentials parameters made an increase in energy values, with the increment of ω causes more increment than others. Exceptions for ρ_i enhancement, it decreases the energy values, although the effect is insignificant.

The behavior of radially DRSO plus Manning Rosen's potentials wave function in 6-dimensional system as a function of r are presented from Figures 3 and 4. Figure 3 presents the effect of orbital quantum numbers n_l against radial ground state wave functions. The graphic shows that in general, the increases of n_l cause the amplitude to become large, together with the wave functions move to the left. The angular quantum number n_{l5} has more influence than others. Exceptional for the increment of n_{l1} it causes the wave amplitude to becomes lower, and the wave function moves to the right. Figure 4 presents the potential parameters against the radial ground state wave functions. The graphic shows that the amplitude of wave function becomes large and move to the left caused by the increase of ω and ρ_i, but becomes lower and move to the right cause by the increase of α, σ, and v_i.

For Figure 5 – Figure 9 it was set $\hbar = \mu = 1, n_r = n_{li} = 1$. It was plotted the partition function $Z(\beta)$ and some thermodynamical properties such as $U(\beta)$, $C(\beta)$, $F(\beta)$ and $S(\beta)$ for various values of frequencies ω and parameters of angular potentials.

It could be observed from Figure 5 that the partition function Z decreases monotonically with increasing of β and some potential parameters ω, α, σ, and v_i. With, the frequency ω from DRSO plus Manning Rosen Potential leads more dominant than some angular parameters α, σ, v_i. Unlike other, increasing of all ρ_i escalates the Z becoming more increase, although it is too small compared to the ω. From Figure 6, for every value of DRSO plus Manning Rosen potential parameters ω, α, σ, v_i and ρ_i it is offered, the vibrational mean energy U decreases monotonically as the increasing of β. Moreover, U is increased when the frequency and other angular parameters α, σ, v_i enlarged. With, the ω parameter leads more dominant than some angular parameters α, σ, v_i. Exceptional for ρ_i, increasing of all ρ_i decreases the Z, although it is too small compared to the ω. Based on Figure 7,

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the specific heat C decreases monotonically as increasing of β and frequencies ω. Meanwhile, this is unique because other angular parameters α, σ, ν_i, and ρ_i are not affecting C. Figure 8 shows that for every value of ω, α, σ, ν_i, and ρ_i it is offered, the vibrational free energy F of the system with DRSO plus Manning-Rosen potential increases monotonically as the increasing of β and some potentials parameters. With, ν_i, and ω lead more dominant than α and σ. Meanwhile, the increase in all ρ_i decreases the F, although it is too small compared to other parameters. From Figure 9, it can be observed that for every value of potential parameters it is offered, the vibrational entropy S of the system decreases monotonically as the increasing of β and ω. It appears that variations on angular parameters α, σ, ν_i and ρ_i does not effect on S.

4. CONCLUSION:

The energy spectrum and ground-state wave function of the 6-dimensional Schrodinger equation governed by the Double Ring-Shape Oscillator (DRSO) plus Manning Rosen potential have been obtained using SUSY QM method. In general, the increment of orbital quantum numbers decreases the energy, but specifically for the increment of n_{l1} cause increases at energy values. Overall, the increment of the potential parameters increases the energy values, with the increment of ω causes more enhancement in energy values compares to the other potentials parameters. The angular quantum numbers and potential parameters have also influence on wave functions. In general, the increment of orbital quantum numbers and the potential parameters causes the wave amplitude to become high, and the wave functions move to the left. Specifically, for the increment at n_{l1}, α, σ, and ρ cause the wave amplitude to become low, and the wave functions move to the right. The energy spectrum equation was used to derive the partition function. This partition function was employed to obtain the thermodynamics properties such as vibrational and vibrationalmean energy U, vibrational specific heat C, vibrational free energy F and vibrational entropy energy S. The vibrational mean energy U and free energy F were increasing as the increased of all potentials parameters, with the ω is more dominant compare to the others. Vibrational specific heat C and vibrational entropy S only affected by the ω, where C and S decreasing as the increment of the ω.

5. ACKNOWLEDGEMENTS:

This research was partly supported by the Mandatory Research Grant of Sebelas Maret University with contract number 452/UN27.21/PN/2020.

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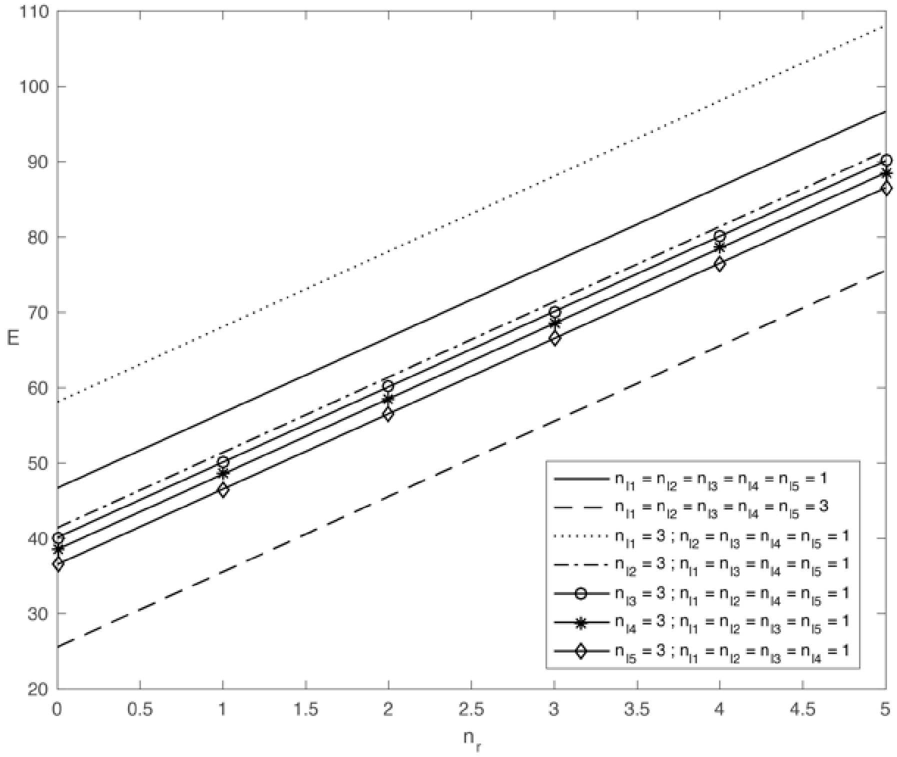

Figure 1. The plot of energy for various n_l
$\hbar = \mu = 1$, $\omega = \alpha = \sigma = \upsilon_i = \rho_i = 5$

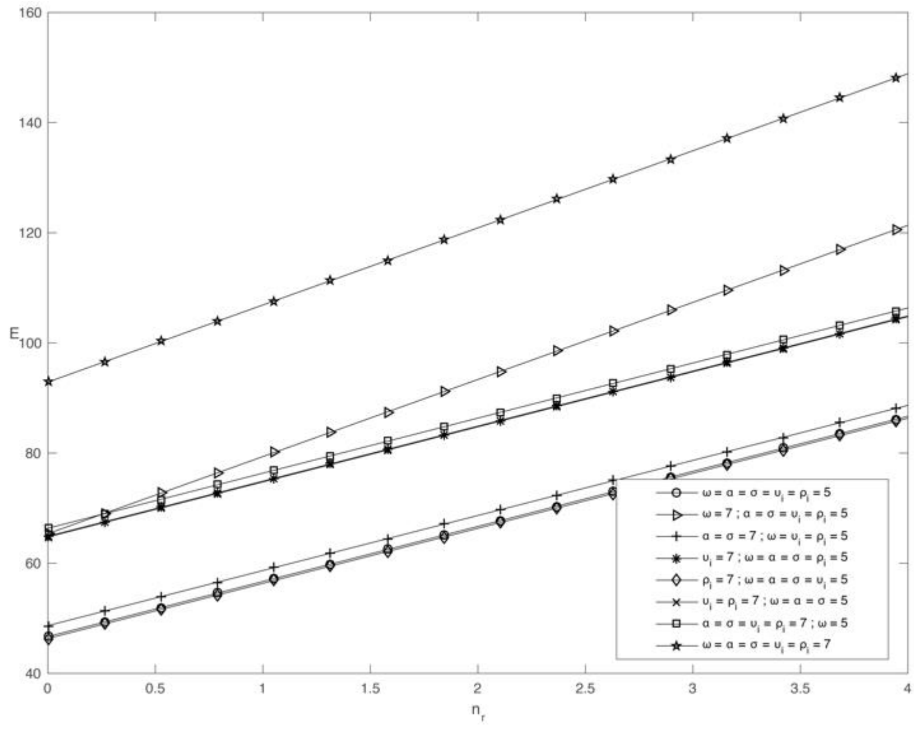

Figure 2. The plot of energy for various potentials parameter
$\hbar = \mu = 1, n_{li} = 1$

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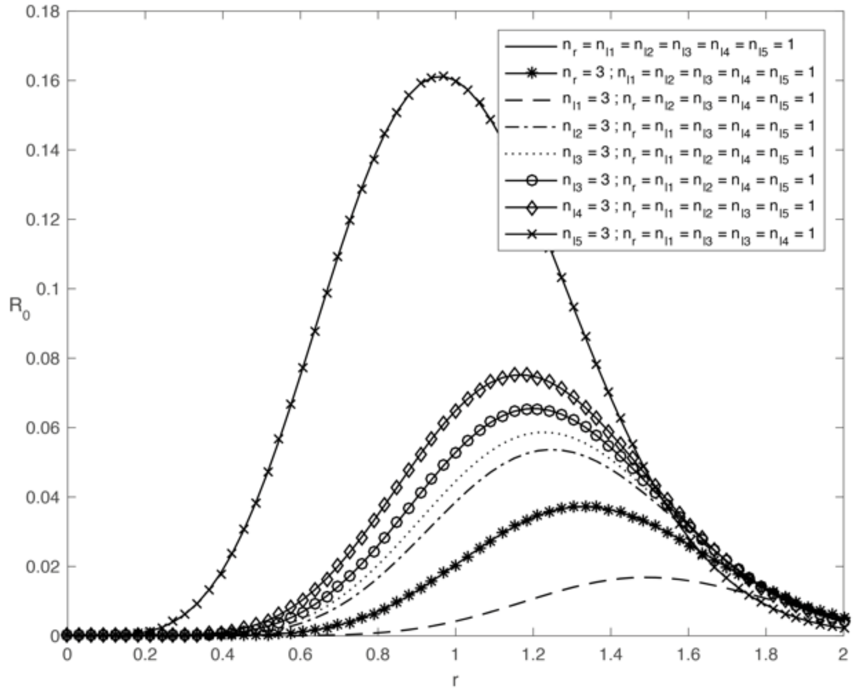

Figure 3. The plot of the groundstate radial wave function R_0 vs r for various n_l
$\hbar = \mu = 1, \omega = \alpha = \sigma = \upsilon_i = \rho_i = 5$

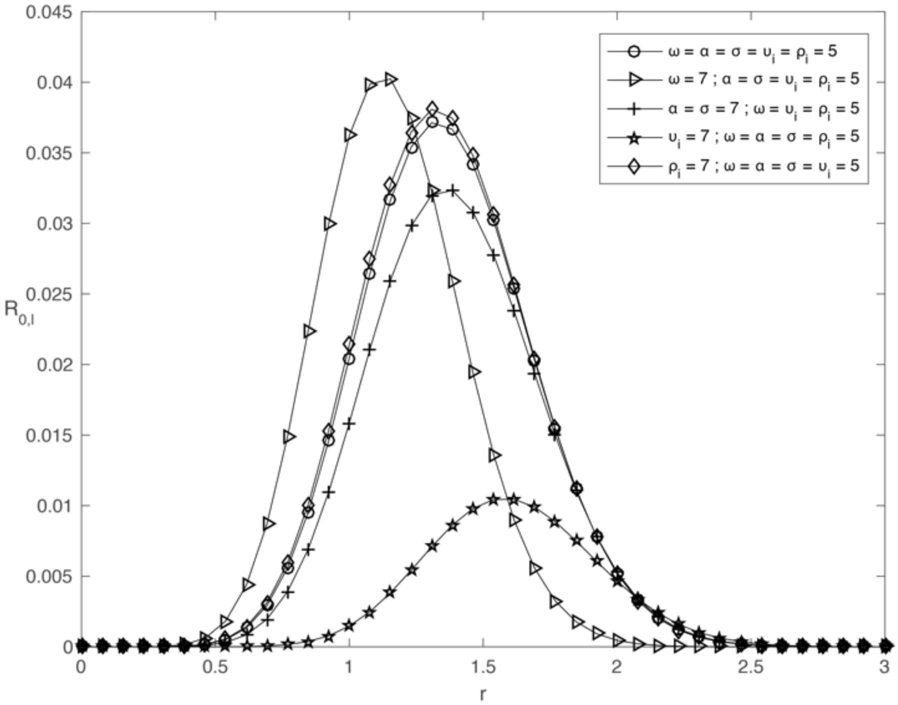

Figure 4. The plot of the groundstate radial wave function R_0 vs r for various potentials parameters
$\hbar = \mu = 1, n_r = n_{li} = 1$

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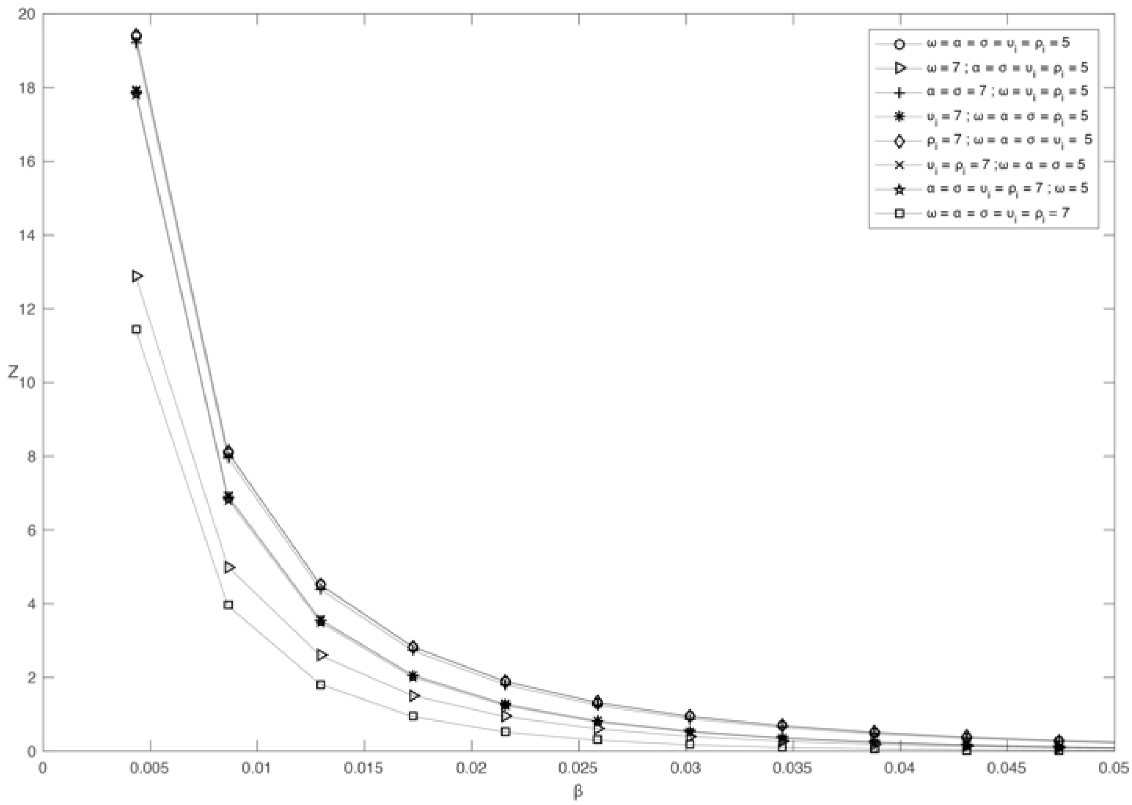

Figure 5. Partition function Z vs β
$\hbar = \mu = 1, n_r = n_{li} = 1$

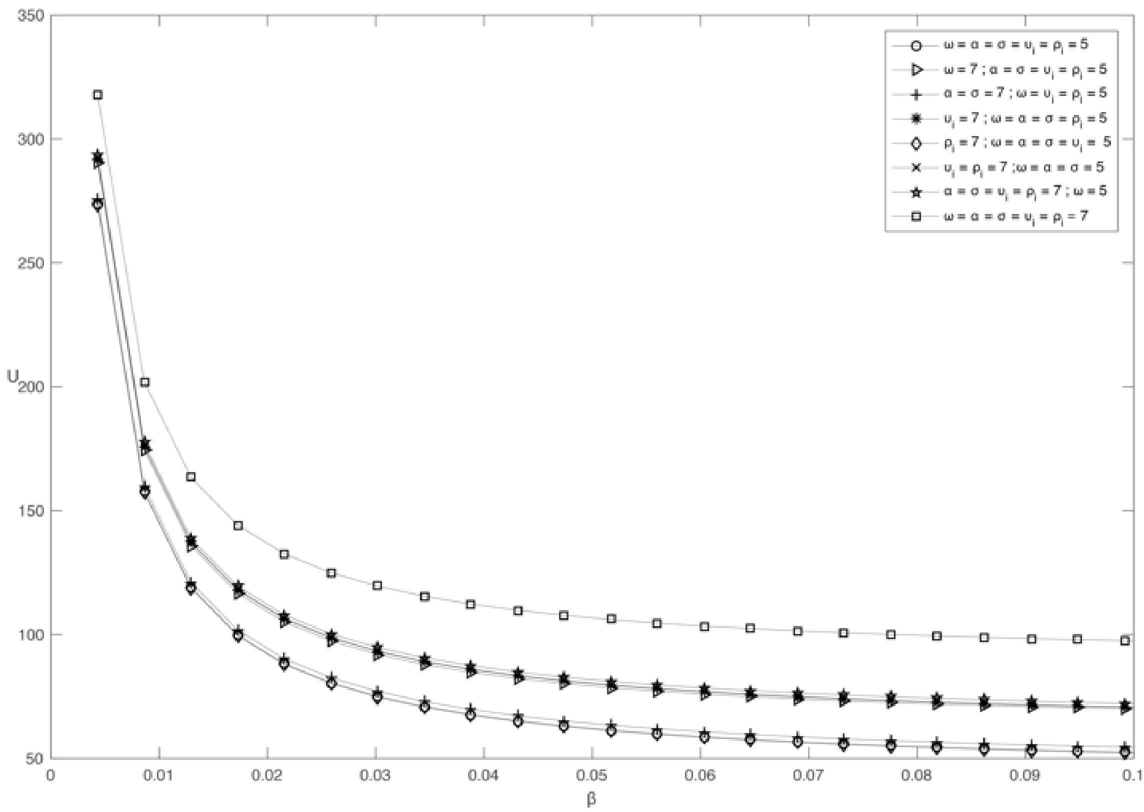

Figure 6. Vibrational mean energy U vs β
$\hbar = \mu = 1, n_r = n_{li} = 1$

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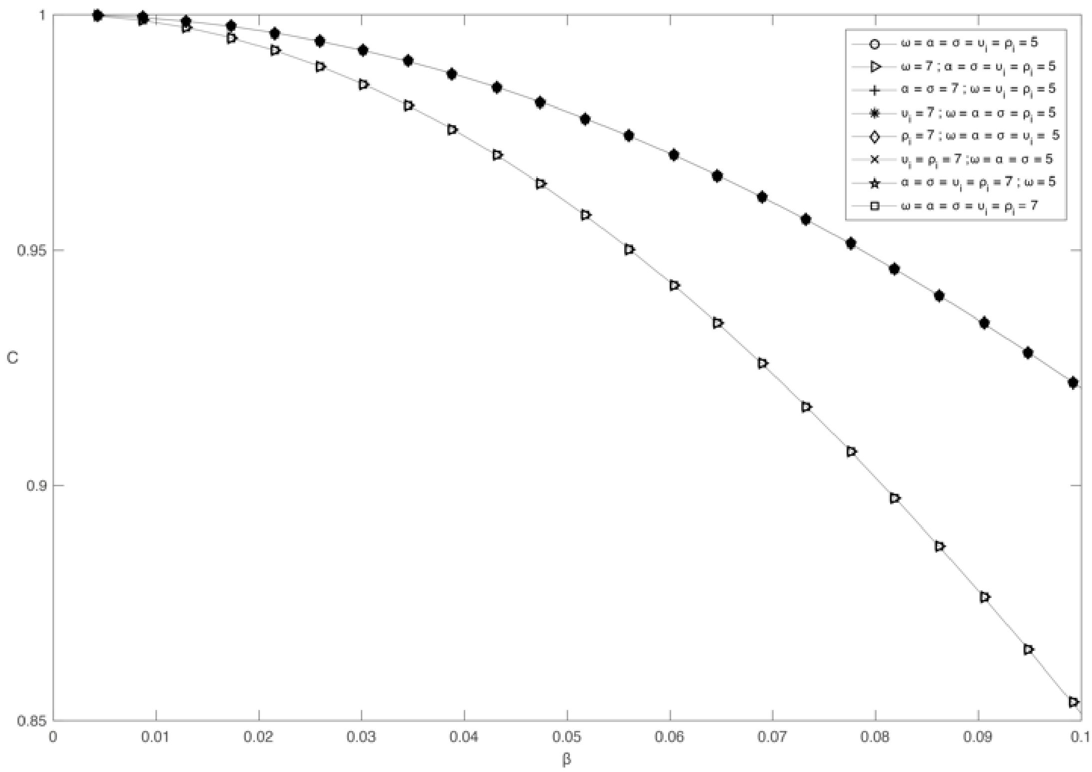

Figure 7. Vibrational specific heat C vs β
$\hbar = \mu = 1, n_r = n_{li} = 1$

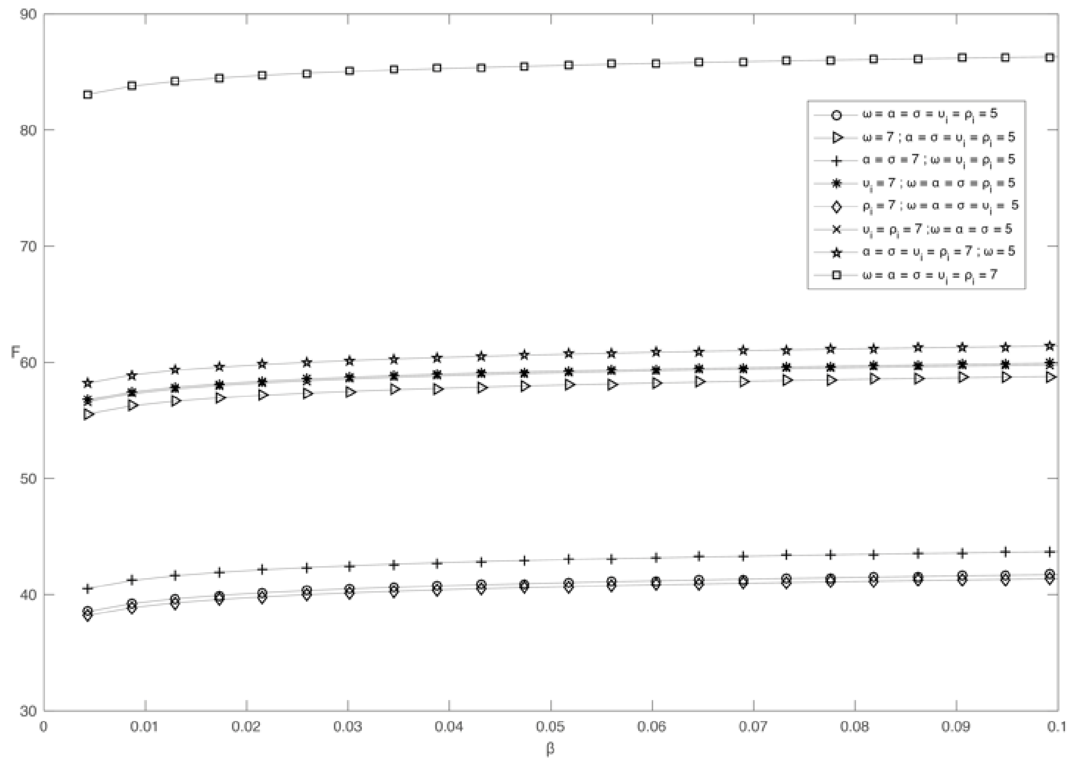

Figure 8. Vibrational free energy F vs β
$\hbar = \mu = 1, n_r = n_{li} = 1$

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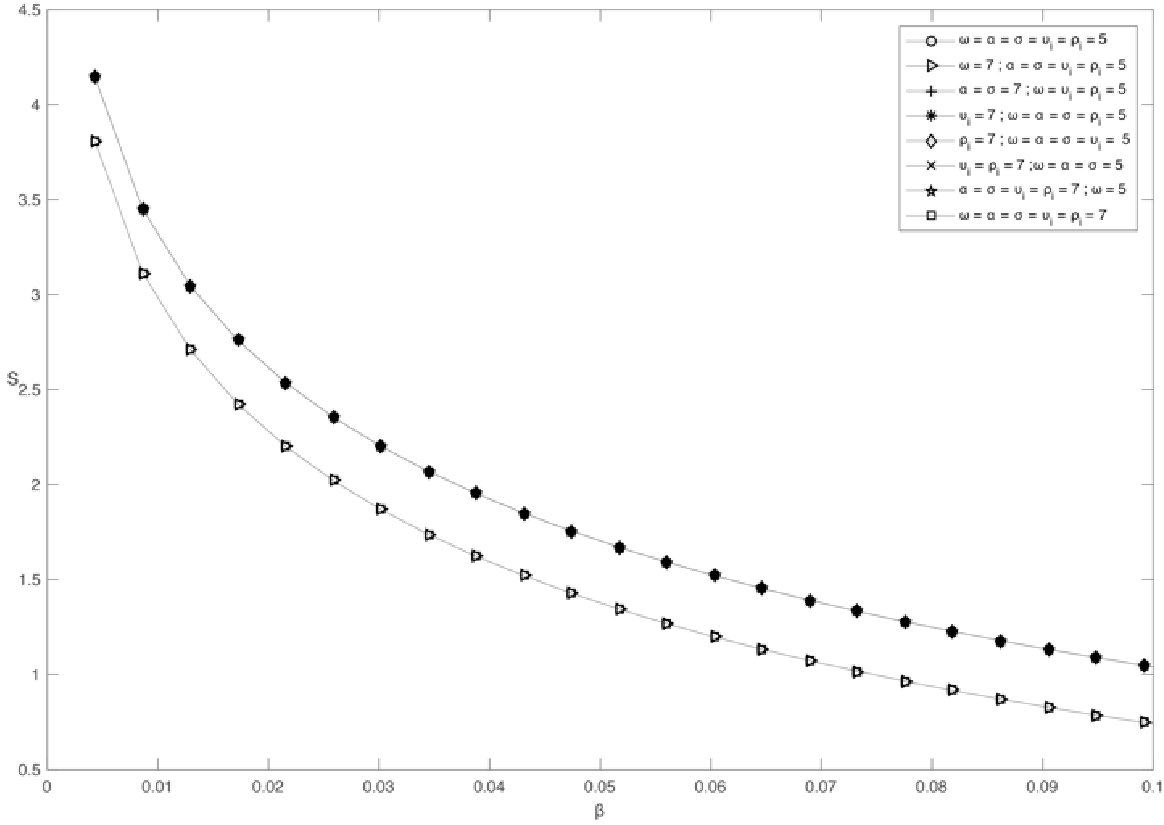

Figure 9. Vibrational entropy S vs β
$\hbar = \mu = 1, n_r = n_{li} = 1$

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